

The University of Chicago

THE SCATTERING OF LIGHT BY LIGHT ACCORDING
TO THE BORN-INFELD THEORY

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE PHYS-
ICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHYSICS

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The Scattering of Light by Light According to the Born-Infeld Theory

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The nonlinearity of the unquantized Born-Infeld field equations leads to the prediction that light should scatter light. The result follows also from the Dirac electrodynamics; however, the natures of the effects are essentially different. Whereas in the quantum electrodynamics a photon interpretation proves tenable, in the Born theory such is not the case. Here the scattering occurs only if the

proper relationship exists between the frequencies and directions of the incident beams, and the scattering takes place only into certain frequencies and directions characteristic of the initial radiation. It is found that the two theories are comparable as to order of magnitude and dependence on the frequency.

A CONSEQUENCE of the linearity of Maxwell's equations is the superposition law for electromagnetic fields, and it has often been pointed out that according to theories based on nonlinear field equations a scattering of light by light should occur. Such an effect should, therefore, be calculable from either the unquantized Born-Infeld¹ theory or from the Dirac quantum electrodynamics. Indeed, it has sometimes been rather loosely stated that the two theories are equivalent in this respect. Although the energy density expressions in them are of the same order of magnitude and differ only in the constants, the predicted scattering may be expected to differ considerably since the one is purely classical, the other essentially quantum mechanical. The prediction as a consequence of the Dirac hole theory has been examined principally by Euler, Kockel and Heisenberg.² Euler has noted that the results of his computations are of a purely quantum mechanical nature. It is the purpose of the present discussion to consider the scattering according to the classical concepts of the Born-Infeld theory.

From Euler's calculations, it follows that according to the Dirac theory, the scattering may be described by the photon theory: two primary photons $h\nu_1$ and $h\nu_2$ collide and form two scattered photons $h\nu_3$ and $h\nu_4$. This result may be reformulated for comparison with the present theory as follows: if at some time, the radiation field consists of two beams of frequency ν_1 and ν_2 , wave number vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, then in the

course of time other (scattered) beams will appear. The ν_3 , $\mathbf{k}_3$ of these beams are restricted by the requirement that there be a ν_4 , $\mathbf{k}_4$ such that

$$\nu_3 + \nu_4 = \nu_1 + \nu_2, \quad \mathbf{k}_3 + \mathbf{k}_4 = \mathbf{k}_1 + \mathbf{k}_2.$$

On the other hand, calculation from a classical nonlinear theory leads to radically different results. Again, if the primary radiation consists of two beams, scattered beams of new frequency and direction (ν_3 , $\mathbf{k}_3$) appear, but these are determined by the equations

$$\nu_3 = n_1\nu_1 + n_2\nu_2, \quad \mathbf{k}_3 = m_1\mathbf{k}_1 + m_2\mathbf{k}_2,$$

the m 's and n 's being small integers whose values are determined by the nonlinear part of the equation. Thus the phenomenon is a sort of "Bragg" reflection, the interference pattern of the two primary waves forming the moving lattice. It is probable that this type of scattering will also occur according to the Dirac theory.

Born's theory of the electromagnetic field leads to equations which in ordinary units take the following form:

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} &= 0, & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{D} &= 0, & \nabla \cdot \mathbf{D} &= 0. \end{aligned} \quad (1)$$

Evidently these would reduce to the equations of Maxwell for a vacuum if $\mathbf{B}$ were equal to $\mathbf{H}$ and $\mathbf{E}$ were equal to $\mathbf{D}$. However, in this theory a vacuum is polarized by the fields, the electric displacement and magnetic induction being obtained from the relations,

$$\mathbf{D} = (\mathbf{E} + \epsilon \mathbf{B} \mathbf{G})/K, \quad \mathbf{H} = (\mathbf{B} - \epsilon \mathbf{E} \mathbf{G})/K, \quad (2)$$

¹ Born and Infeld, *Proc. Roy. Soc.* **144**, 424 (1934).

² Halpern, *Phys. Rev.* **45**, 855 (1934); Euler and Kockel, *Naturwiss.* **12**, 246 (1935); Heisenberg and Euler, *Zeits. f. Physik* **98**, 714 (1936); Euler, *Ann. d. Physik* **26**, 398 (1936).

where $K = (1 + \epsilon F - \epsilon^2 G^2)^{1/2}$, $\epsilon = \text{constant}$,
 $F = B^2 - E^2$, $G = \mathbf{B} \cdot \mathbf{E}$.

The order of magnitude³ of ϵ is 10^{-29} , and therefore for most cases the results arrived at will differ very little from those obtained by the more usual equations. Hence it is to be expected that the scattering due to the interaction of light beams will be very small.

The primary quantities describing the field are chosen to be $\mathbf{B}$ and $\mathbf{E}$. For the present purpose the second set (1) is written in another form involving the virtual currents and charges produced by the polarization of space:

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{j}, \quad \nabla \cdot \mathbf{E} = 4\pi \rho, \quad (3)$$

where $4\pi \mathbf{j} = \nabla \times (\mathbf{B} - \mathbf{H}) + \frac{1}{c} \frac{\partial (\mathbf{D} - \mathbf{E})}{\partial t}$,
 $4\pi \rho = -\nabla \cdot (\mathbf{D} - \mathbf{E}). \quad (4)$

Introduction of the potentials $\mathbf{A}$ and ϕ in the customary manner and the usual Maxwell auxiliary condition reduces the four field equations to:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -4\pi \mathbf{j}, \quad \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho. \quad (5)$$

Since ϵ is a very small parameter, it is possible to develop quantities involving it in a power series of the form:

$$X = X^{(0)} + \epsilon X^{(1)} + \epsilon^2 X^{(2)} + \dots, \quad (6)$$

where X represents any of the quantities entering the theory. Expanding $\mathbf{A}$ and $\mathbf{j}$, ϕ and ρ in this manner and equating coefficients of ϵ^n , from Eq. (4) we get

$$\begin{aligned} \nabla^2 \mathbf{A}^{(n)} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}^{(n)}}{\partial t^2} &= -4\pi \mathbf{j}^{(n)}, \\ \nabla^2 \phi^{(n)} - \frac{1}{c^2} \frac{\partial^2 \phi^{(n)}}{\partial t^2} &= -4\pi \rho^{(n)}. \end{aligned} \quad (7)$$

In particular $j^{(0)}$ and $\rho^{(0)}$ are identically zero. It will be supposed that $\mathbf{A}^{(n)}$, $\partial \mathbf{A}^{(n)} / \partial t$, $\phi^{(n)}$, and $\partial \phi^{(n)} / \partial t$ ($n \neq 0$) are zero at all points in space at $t=0$, while $\mathbf{A}^{(0)}$, $\phi^{(0)}$ are known functions at the initial time.

³ This is for the corrected value which takes account of the electron spin; Born and Schrödinger, *Nature* **135**, 342 (1935).

It is convenient to suppose that the radiation under consideration is confined in a unit cube, and that the fields have the same values at opposite points of the boundary of the cube.⁴ Then $\mathbf{A}$ and ϕ may be expanded in series:

$$\mathbf{A} = \sum a_\lambda \mathbf{A}_\lambda + \sum a_\sigma \mathbf{A}_\sigma, \quad \phi = \sum b_\sigma \phi_\sigma, \quad (8)$$

where the a 's and b 's are functions of time alone and

$$\begin{aligned} \mathbf{A}_\lambda &= \mathbf{e}_\lambda (4\pi c^2)^{1/2} \exp(i\mathbf{k}_\lambda \cdot \mathbf{r}), \\ \mathbf{A}_\sigma &= \mathbf{f}_\sigma (4\pi c^2)^{1/2} \exp(i\mathbf{k}_\sigma \cdot \mathbf{r}), \\ \phi &= (4\pi c^2)^{1/2} \exp(i\mathbf{k}_\sigma \cdot \mathbf{r}), \end{aligned} \quad (9)$$

$\mathbf{e}_\sigma$ being a unit vector in the direction of polarization, and $\mathbf{f}_\sigma$ a unit vector in the direction of propagation. These satisfy

$$\nabla^2 \mathbf{A}_\lambda - \nu_\lambda^2 / c^2 \mathbf{A}_\lambda = 0, \quad \nabla^2 \mathbf{A}_\sigma - \nu_\sigma^2 / c^2 \mathbf{A}_\sigma = 0, \quad (10)$$

$$\nabla^2 \phi_\sigma - \nu_\sigma^2 / c^2 \phi_\sigma = 0,$$

$$\text{and} \quad \nabla \cdot \mathbf{A}_\lambda = 0, \quad \nabla \cdot \mathbf{A}_\sigma = k_\sigma \phi_\sigma,$$

$$\nabla \phi_\sigma = k_\sigma \mathbf{A}_\sigma, \quad \nabla \times \mathbf{A}_\lambda = \mathbf{k}_\lambda \times \mathbf{A}_\lambda.$$

Thus $\mathbf{A}_\lambda$ belongs to the transverse part of the vector potential, and $\mathbf{A}_\sigma$ to the longitudinal part. These quantities are normalized so that

$$\int \mathbf{A}_\lambda \cdot \mathbf{A}_{\lambda'}^* d\tau = 4\pi c^2 \delta_{\lambda\lambda'}, \quad \int \mathbf{A}_\sigma \cdot \mathbf{A}_{\sigma'}^* d\tau = 4\pi c^2 \delta_{\sigma\sigma'}, \quad (11)$$

$$\int \phi_\sigma \phi_{\sigma'}^* d\tau = 4\pi c^2 \delta_{\sigma\sigma'}, \quad \int \mathbf{A}_\lambda \cdot \mathbf{A}_\sigma^* d\tau = 0.$$

In the expanded form of $\mathbf{j}$ and ρ will occur terms in $(\mathbf{D}^{(n)} - \mathbf{E}^{(n)})$ and $(\mathbf{B}^{(n)} - \mathbf{H}^{(n)})$ (see Eq. (4)), the first two pairs of which by (2) are found to be:

$$\begin{aligned} \mathbf{D}^{(1)} - \mathbf{E}^{(1)} &= \mathbf{B}^{(0)} G^{(0)} - \mathbf{E}^{(0)} F^{(0)} / 2, \\ \mathbf{B}^{(1)} - \mathbf{H}^{(1)} &= \mathbf{E}^{(0)} G^{(0)} + \mathbf{B}^{(0)} F^{(0)} / 2, \\ \mathbf{D}^{(2)} - \mathbf{E}^{(2)} &= \mathbf{B}^{(0)} G^{(1)} + \mathbf{B}^{(1)} G^{(0)} - \mathbf{E}^{(0)} F^{(1)} / 2 \\ &\quad - \mathbf{E}^{(1)} F^{(0)} / 2 + \frac{3}{8} \mathbf{E}^{(0)} F^{(0)} F^{(0)} \\ &\quad + \mathbf{E}^{(0)} G^{(0)} G^{(0)} / 2 - \mathbf{B}^{(0)} F^{(0)} G^{(0)} / 2, \\ \mathbf{B}^{(2)} - \mathbf{H}^{(2)} &= \mathbf{E}^{(0)} G^{(1)} + \mathbf{E}^{(1)} G^{(0)} + \mathbf{B}^{(0)} F^{(1)} / 2 \\ &\quad + \mathbf{B}^{(1)} F^{(0)} / 2 - \frac{3}{8} \mathbf{B}^{(0)} F^{(0)} F^{(0)} \\ &\quad - \mathbf{B}^{(0)} G^{(0)} G^{(0)} / 2 - \mathbf{E}^{(0)} F^{(0)} G^{(0)} / 2. \end{aligned} \quad (12)$$

⁴ Cf., for instance, Heitler, *Quantum Theory of Radiation*, Chap. I, Sec. 6.

It may, therefore, be seen that $\mathbf{j}^{(1)}$ is a function involving terms of degree 3 in $\mathbf{A}^{(0)}$ and $\phi^{(0)}$, and that $\mathbf{j}^{(2)}$ is a function involving terms of degree 5 in $\mathbf{A}^{(0)}$ and $\phi^{(0)}$ and terms of degree 2 in $\mathbf{A}^{(0)}$ and $\phi^{(0)}$ times first degree in $\mathbf{A}^{(1)}$ or $\phi^{(1)}$.

In harmony with Eq. (8), let

$$\mathbf{A}_\lambda^{(0)} = \sum \mathbf{e}_\lambda (4\pi c^2)^{1/2} [\epsilon_{\lambda\lambda} \exp(i\nu_\lambda t) + \epsilon_{-\lambda\lambda} \exp(-i\nu_\lambda t)] \exp(i\mathbf{k}_\lambda \cdot \mathbf{r}), \quad (13)$$

while, for simplicity, the initial longitudinal component will be supposed to be zero. The summation is to cover both positive and negative sign before $\mathbf{k}_\lambda$, the coefficients of $\exp(i\mathbf{k} \cdot \mathbf{r})$ and $\exp(-i\mathbf{k} \cdot \mathbf{r})$ being conjugates. In the following discussion (13) will be restricted to the case of two initial beams. By consideration of (3), (12) and (13) it is seen that $\mathbf{j}^{(1)}$ for such a case is of the form:

$$\begin{aligned} \mathbf{j}^{(1)} &= \sum_{nm} (\mathbf{e}_m (4\pi c^2)^{1/2}) j_{nm}^{(1)} \exp i[(m_1 \mathbf{k}_1 + m_2 \mathbf{k}_2) \cdot \mathbf{r} \\ &\quad + (n_1 \nu_1 + n_2 \nu_2) t], \quad (14) \\ &= \sum_{0m} (\mathbf{e}_m (4\pi c^2)^{1/2}) j_{0m}^{(1)} \exp i(\mathbf{k}_m \cdot \mathbf{r} + \nu_0 t). \end{aligned}$$

where m_1 and m_2 may take any integral value between -3 and $+3$ provided $|m_1 - m_2| = 1, 3$. The corresponding relation holds for the n 's with the additional restriction that n_1 and m_1 (or n_2 , m_2) must both be even or both odd. The equation for the first order terms of the potential may now, with the help of (8), (9), (11) and (14), be reduced to

$$\frac{d^2 a_\lambda^{(1)}}{dt^2} + \nu_\lambda^2 a_\lambda^{(1)} = 4\pi c^2 \sum_0 j_{0\lambda}^{(1)} \exp(i\nu_0 t), \quad (15)$$

the summation being only over those ν_0 permitted for $\mathbf{k}_\lambda$ by the conditions on Eq. (14). A similar type of equation results for the longitudinal part of the vector potential and for the scalar potential, but for the present only the transverse part will be considered. The solution of this equation will consist of small oscillating terms and terms whose amplitude increases with time (i.e., resonance terms). In order that a term in the solution be of increasing amplitude, it is necessary that

$$c|\mathbf{k}_\lambda| = \pm \nu_0$$

$$\text{i.e.,} \quad c|m_1 \mathbf{k}_1 + m_2 \mathbf{k}_2| = \pm (n_1 \nu_1 + n_2 \nu_2). \quad (16)$$

For given magnitudes of ν_1 and ν_2 this may be fulfilled for those permitted values of m and n only if the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ satisfies the following:

$$\begin{aligned} &\pm \cos \theta \\ &= \frac{(m_1^2 - n_1^2) \nu_1^2 + (m_2^2 - n_2^2) \nu_2^2 - 2n_1 n_2 \nu_1 \nu_2}{2m_1 m_2 \nu_1 \nu_2}. \quad (17) \end{aligned}$$

By a similar consideration and the use of the solution of (15),

$$\begin{aligned} \mathbf{j}^{(2)} &= \sum_{nm} (\mathbf{e}_m (4\pi c^2)^{1/2}) (j_{nm}^{(2)} - k_{nm}^{(2)} t) \\ &\quad \times \exp i[(m_1 \mathbf{k}_1 + m_2 \mathbf{k}_2) \cdot \mathbf{r} + (n_1 \nu_1 + n_2 \nu_2) t] \\ &= \sum_{0m} (\mathbf{e}_m (4\pi c^2)^{1/2}) (j_{0m}^{(2)} - k_{0m}^{(2)} t) \\ &\quad \times \exp i(\mathbf{k}_m \cdot \mathbf{r} + \nu_0 t), \quad (14') \end{aligned}$$

where the m 's and n 's may take any integral value between -5 and $+5$ provided $|m_1 - m_2| = 1, 3, 5$, and a similar relation for the n 's, and again n_1, m_1 are both even or both odd. Corresponding to (15), the equation for the second order terms of the transverse potential becomes

$$\begin{aligned} d^2 a_\lambda^{(2)} / dt^2 + \nu_\lambda^2 a_\lambda^{(2)} &= 4\pi c^2 \sum_0 (j_{0\lambda}^{(2)} - k_{0\lambda}^{(2)} t) \\ &\quad \times \exp(i\nu_0 t), \quad (15') \end{aligned}$$

the solution of which will also consist of small terms and terms whose amplitude increases with time. Again an equation like (17) determines the angles at which two beams of given frequency must intersect in order to obtain resonance terms.

In order to discuss the scattering, it is necessary to have a definition of intensity. It will be supposed that the time average of

$$\frac{1}{2} [|da_\lambda^{(2)} / dt|^2 + \nu_\lambda^2 |a_\lambda^{(2)}|^2] \quad (18)$$

is the intensity of radiation of wave number $\mathbf{k}_\lambda$. In the Maxwell theory, the justification for this is that the total energy can be expressed as the sum of such terms, plus contributions from the longitudinal components. In the Dirac theory the justification is to be found in the photon interpretation of the process of quantization. In the Born-Infeld theory, the definition of intensity as a function of the frequency is not clear; in order to keep as close as possible to the other theories, expression (18) will be assumed to be

the "intensity," but, because of the nonlinear nature of the equations, it cannot be identified with "energy."

In the preceding discussion only one oscillator of frequency ν_λ has been considered, it may be shown however that there are $\nu^2 d\nu d\Omega / (2\pi c)^3$ of these natural oscillators in the frequency range having directions within the solid angle $d\Omega$. This factor must therefore multiply the equation. It is desired to average this over a small frequency range in order to eliminate the accidental interference effects which arise from the initial coherence of the vibrations. Hence the desired intensity is given by

$$I_\lambda = (1/T) \int_0^T I_\lambda^v dt, \quad (19)$$

where

$$I_\lambda^v = \int_{\nu-\Delta\nu}^{\nu+\Delta\nu} \left(\frac{\nu^2 d\nu}{(2\pi c)^3} \left[\left| \frac{da_\lambda}{dt} \right|^2 + \nu_\lambda^2 |a_\lambda|^2 \right] \right).$$

Expansion of (18) yields

$$\begin{aligned} & \frac{1}{2} \left[\left| \frac{da_\lambda^{(0)}}{dt} \right|^2 + \nu_\lambda^2 |a_\lambda^{(0)}|^2 \right] + \epsilon \left[\left| \frac{da_\lambda^{(1)}}{dt} \frac{da_\lambda^{(0)}}{dt} \right| \right. \\ & + \nu_\lambda^2 |a_\lambda^{(1)} a_\lambda^{(0)}|^2 \left. \right] + \epsilon^2 \left[\frac{1}{2} \left(\left| \frac{da_\lambda^{(1)}}{dt} \right|^2 + \nu_\lambda^2 |a_\lambda^{(1)}|^2 \right) \right. \\ & + \left(\left| \frac{da_\lambda^{(2)}}{dt} \frac{da_\lambda^{(0)}}{dt} \right| + \nu_\lambda^2 |a_\lambda^{(2)} a_\lambda^{(0)}|^2 \right) \left. \right] + \dots \quad (20) \end{aligned}$$

Upon substitution of the values of the a 's from (13) and from the solutions of (15) and (15'), and carrying out the averaging processes indicated in (19); it is found that the first term is

the Maxwell energy, the second term gives no contribution which increases with time, while the third term gives the scattered intensity:

$$S_\lambda = \{ (c\nu_\lambda^2/2) [|j_{\lambda\lambda}^{(1)}|^2 + |j_{-\lambda\lambda}^{(1)}|^2] + c\nu_\lambda [|ic_{\lambda\lambda}k_{\lambda\lambda}| + |ic_{-\lambda\lambda}k_{-\lambda\lambda}|] \} \epsilon^2 t. \quad (21)$$

If the functions j , c and k are expressed in terms of the initial field strengths, it is found that the ratio of scattered intensity per unit time to the incident intensity is of the order of magnitude $\nu^4 (E^{(0)})^4 \epsilon^2 / 2^{10} \pi^2 c^3$. For the sake of comparison with the quantum-mechanical results, put $(E^{(0)})^2 / 4\pi = nh\nu / 2\pi$, where n is number of photons passing through unit cross section per unit time. Then the ratio is equivalent to $\nu^6 (hn)^2 / 2^8 \pi^2 c^3$ or $10^{-88} / \lambda^6$ (when n is of order 10^0). The quantum-mechanical deduction of Euler gives the transition probability for two colliding photons to be

$$\frac{8\pi}{5 \cdot 3^4} [3 + \cos^2 \phi]^2 (e^2 / mc^2)^2 (h/mc)^4 \frac{1}{\lambda^6},$$

which is of the order $10^{-89} / \lambda^6$. However, although the magnitude of the scattering in the two cases is comparable, it should be emphasized that the nature of the phenomenon is fundamentally different; for while the Dirac theory yields a probability of scattering into all angles ϕ , the nonlinear addition to the classical field equations gives scattering only if the angular relationship of the beams is correct, and the scattering is only into a few characteristic directions.

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